Problem Set 8

Chapter 6 and extra.

Note: Z1, Z2 etc. refer to my own problems, which may be very close to what is in the text.

Z1. Suppose that $T \in \mathcal{L}(\mathbb{C}^4)$ has just two eigenvalues, -2 and 3. In addition $\dim \operatorname{null}(T+2I) + \dim \operatorname{null}(T-3I) = 2$. Answer true or false.

- $\mathbb{C}^4$ must have a basis of eigenvectors of T.
- $\mathbb{C}^4$ might have a basis of eigenvectors of T.
- $\mathbb{C}^4 = \operatorname{null}(T+2I)^2 \oplus \operatorname{null}(T-3I)^2$.
- $\mathbb{C}^4$ has a basis (v_1, v_2, v_3, v_4) such that

$$\operatorname{Mat}(T, (v_1, v_2, v_3, v_4)) = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

- $T + 2I$ is a nilpotent operator.
- $T^2 - T - 6I$ is a nilpotent operator.

Z2. In $\mathbb{R}^3$, let $v_1 = (1, -1, 1)$, $v_2 = (1, 1, 1)$ and $v_3 = (1, 0, 2)$. Apply the Gramm-Schmidt algorithm to (v_1, v_2, v_3) to obtain an orthonormal basis for $\mathbb{R}^3$. Let $U = \operatorname{span}(v_1, v_2)$. Compute $P_U(1, 1, 1)$, $P_U(1, 0, 1)$ and $P_{U^\perp} v_3$.

Z3. (Compare with problem 10). On $\mathbb{R}[x]$, consider the inner product given by

$$\langle p, q \rangle = \frac{1}{2} \int_{-1}^{1} p(x) q(x)\, dx.$$

Apply the Gram-Schmidt procedure to $(1, x, x^2, x^3)$ to produce an orthonormal basis of $\mathbb{R}_3[x]$.

Z4. Compute an orthonormal basis of $\mathbb{R}_3[x]$ (using the same inner product as in problem Z3) such that D, the derivative operator, has an upper triangular (UT) matrix with respect to this basis.

2. Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbb{F}$.

3. Prove that

$$\Big(\sum_{j=1}^{n} a_j b_j\Big)^2 \leq \Big(\sum_{j=1}^{n} j {a_j}^2\Big)\Big(\sum_{j=1}^{n} \frac{{b_j}^2}{j}\Big)$$

for all real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

11. What happens if the Gram-Schmidt procedure is applied to a list of vectors that is not linearly independent?

    COMMENT: Not assigned. I discussed this in class.

13. Suppose $(e_1, \ldots, e_m)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \ldots, e_m)$.

18. Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|$$

for every $v \in V$, then P is an orthogonal projection.

19. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.