Problem Set 7

Chapters 4, 5 and extra.

Note: Z1, Z2 etc. refer to my own problems, which may be very close to what is in the text.

Z1. Review and relevant to the next problem. Suppose that $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, V)$, where $\dim V = n$ and $\dim W = m$.

- If $n > m$, prove that ST is not invertible.
- Consider the 3×3 matrix

$$A = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} & a_{1,1}b_{1,3} + a_{1,2}b_{2,3} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} & a_{2,1}b_{1,3} + a_{2,2}b_{2,3} \\ a_{3,1}b_{1,1} + a_{3,2}b_{2,1} & a_{3,1}b_{1,2} + a_{3,2}b_{2,2} & a_{3,1}b_{1,3} + a_{3,2}b_{2,3} \end{bmatrix}$$

  What is the maximum possible rank of A?

Z2. Let $T \in \mathcal{L}(\mathbb{F}^3, \mathbb{F}^2)$ and $S \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ be defined by $T(x, y, z) = (x - y + z, x + y - z)$ and $S(x, y) = (x + 2y, x + y, x + 4y)$. Then $TS \in \mathcal{L}(\mathbb{F}^2)$ and $ST \in \mathcal{L}(\mathbb{F}^3)$.

- Write down an eigenvalue of ST without performing any computations. (Hint: Consider the previous problem.)
- Compute $\mathrm{Mat}(TS)$ with respect to the standard basis in $\mathbb{F}^2$.
- Let $v = (1, 0)$. Compute the smallest $j > 1$ such that $(TS)^j \in \mathrm{span}(v, \ldots, (TS)^{j-1}v)$.
- Compute a polynomial $p(x)$ of lowest degree such that $p(T)v = 0$.
- From this polynomial, compute the eigenvalues of TS.
- Using the eigenvalues of TS together with computed values of $(TS)^i v$, compute an eigenvector for each eigenvalue.
- Does $\mathbb{F}^2$ have a basis of eigenvalues of TS?

Z3. Suppose that $T \in \mathcal{L}(\mathbb{R}^3)$ has eigenvalues $-1, 0$ and 2. Furthermore, you are given $T(1, 0, 0) = (1, 1, 0)$ and $T(1, 1, 0) = (1, 1, 1)$. Compute eigenvectors corresponding to each eigenvalue.

14. Not assigned. Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Prove that if $p \in \mathbb{F}[x])$ is a polynomial, then

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

SOLUTION: Note that $(STS^{-1})^2 = (STS^{-1})(STS^{-1}) = (ST)(S^{-1}S)(TS^{-1}) = (ST)I(TS^{-1}) = ST^2S^{-1}$. If $(STS)^n = ST^nS^{-1}$, then $(STS^{-1})^{n+1} = (STS^{-1})^n(STS^{-1}) = (ST^nS^{-1})(STS^{-1}) = (ST^n)(SS^{-1})(TS^{-1}) = ST^{n+1}S$. Thus, by induction, $(STS^{-1})^n = ST^nS^{-1}$ for all $n >= 0$. If $p(x) = \sum_{i=0}^{m} a_i x^i$, then

$$\begin{aligned} Sp(T)S^{-1} &= S(\sum_{i=0}^{m} a_i T^i)S^{-1} \\ &= \sum_{i=0}^{m} a_i S T^i S^{-1} \\ &= \sum_{i=0}^{m} a_i (STS^{-1})^i \\ &= p(STS^{-1}). \end{aligned}$$

15. Suppose $\mathbb{F} = \mathbb{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbb{C})$, and $a \in \mathbb{C}$. Prove that a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

18. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

19. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

20. Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

21. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$. Note how this is related to a problem assigned in the previous homework. That is, if $P^2 = P$, then the eigenvalues of P are either 0 or 1. This problem is also related to problem 22.