Problem Set 5

Prep for Test1

Note: The problems are labeled T1, T2 and so on. No (obvious) relation to text.

T1. Suppose that U_1, U_2 and U_3 are subspaces of $\mathbb{F}^{20}$ such that $\mathbb{F}^{20} = U_1 + U_2 + U_3$ and $\dim U_3 - \dim U2 = \dim U_2 - \dim U_1 = 1$. Can the sum be direct?

T2. Suppose that U_1, U_2 and U_3 are non-zero subspaces of $\mathbb{F}^n$ such that $\mathbb{F}^n = U_1 \oplus U_2 \oplus U_3$ and $\dim U_3 - \dim U2 = \dim U_2 - \dim U_1 = 2$. What is the smallest value of n for which this could be true?

T3. In $\mathbb{R}^3$, let v_1, v_2, v_3, v_4, v_5 and v_6 be $(1, 2, 3)$, $(-1/6, -1/3, -1/2)$, $(4, 5, 6)$, $(7, 8, 9)$, $(\sqrt{2}, 13, -2)$ and $(-17, 31, 11)$, respectively. Reduce this list to a linearly independent list by finding the smallest index $j > 1$ such that $v_j \in \text{span}(v_1, \ldots v_{j-1})$ and discarding v_j. Then find the next smallest index with the same property and so on until the remaining vectors are linearly independent. What is span of these vectors?

T4. Suppose that $T \in \mathcal{L}(\mathbb{F}^8, \mathbb{F}^6)$. Prove that there must be two non-zero linearly independent vectors $v_1 = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ and $v_2 = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$ in $\mathbb{F}^8$ such that $T(3v_1 + 5v_2) = 0$ and $T(7v_1 - 2v_2) = 0$.

T5. Suppose that $T \in \mathcal{L}(\mathbb{F}^8, \mathbb{F}^6)$ and that if $Tv_1 = Tv_2 = Tv_3 = 0$ for three vectors $v_1, v_2, v_3 \in \mathbb{F}^8$, then (at least) one of these vectors must be in the span of the other two. Prove that there must be some vector $v \in \mathbb{F}^8$ such that $Tv = (1, 2, 3, 4, 5, 6)$.

T6. Suppose that $p_1(x), p_2(x), \ldots, p_m(x)$ are m polynomials with complex coefficients and that they all have different degrees. $m > 1$.
Let $V = \text{span}(p_1(x), p_2(x), \ldots, p_m(x)$. Which of the following are true?

1. For some $n \geq 0$, $\mathbb{C}_n[x] \subseteq V$.
2. For some $n \geq 0$, $V \subseteq \mathbb{C}_n[x]$.
3. $\dim V = m$.

T7. In the vector space of real valued functions of $\mathbb{R}$, $\{f : \mathbb{R} \to \mathbb{R}\}$, consider the subspace $V = \text{span}(e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x})$. If $\lambda_i = \lambda_j$ for some $i \neq j$, then clearly the list is linearly dependent, since if $i < j$ then $e^{\lambda_j x}$ is in the span of the previous vectors. However, if all values of λ_i are distinct, prove that these functions must be linearly independent.

Hint: If they are linearly dependent, then you can write one exponential function as a linear combination of the others. Consider what happens as $x \to \infty$.

T8. Suppose that $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V)W$.

$$U \overset{T}{\longrightarrow} V \overset{S}{\longrightarrow} W$$

1. Suppose that ST is invertible. This means that $\dim U = \dim W = n$. Let $m = \dim V$. What can we say about m? Can m be less than n or greater than n?
2. Suppose that ST is injective. Does S have to be injective?
3. Suppose that ST is surjective. Does T have to be surjective?

T9. Suppose that $p_1(x), p_2(x), \ldots, p_m(x)$ are m polynomials with real coefficients. Assume that $m > 1$ and that each polynomial has degree $< m$. Let $V = \text{span}(p_1(x), p_2(x), \ldots, p_m(x))$. Furthermore, $p_i(\pm 1) = 0$ for all of these polynomials. What is the maximum possible dimension of V? Give an example of polynomials that satisfy the above conditions and span a subspace of maximum dimension for $m = 4$.

T10. Consider the linear equations

$$\begin{aligned} 8x_1 - 3x_2 + 13x_3 &= 18 \\ 7x_1 + x_2 - 4x_3 &= 4 \end{aligned}$$

for $(x_1, x_2, x_3) \in \mathbb{R}^3$. Observe that $(x_1, x_2, x_3) = (1, 1, 1)$ is a solution. Answer true or false to the following.

1. There are no other solutions.
2. The set of all solutions is a subspace of $\mathbb{R}^3$.
3. The set of all solutions is a line.
4. The set of all solutions is a plane.

T11. Suppose that $T \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3)$ and that

$$\mathrm{Mat}(T) = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

with respect to the standard basis. Compute $\dim \operatorname{null} T$. Is T surjective? What is the canonical matrix of this linear map?