Problem Set 4

Chapter 3

Note: Z1, Z2 etc. refer to my own problems, which may be very close to what is in the text.

Z1. Let $T \in \mathcal{L}(V, W)$, and suppose that $\dim V = n$ and $\dim W = m$. From the derivation of $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$ and what I talked about in class, there is a basis in V and a basis in W such that the matrix of T, $(a_{i,j})_{m \times n}$, has the form

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ m \end{array} \quad \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & \dots & k & \dots & n \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{array}$$

A is an $m \times n$ matrix. There is a $k \times k$ identity matrix in the upper left corner. All other entries are 0. $k = \dim \operatorname{range} T$.

- If $k < \min\{m, n\}$, is T surjective? Is T injective?
- If $k = \min\{m, n\}$, is T surjective? Is T injective?

Z2. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by $T(x, y) = (3x - y, y - x)$. What is the matrix for T using the standard basis in $\mathbb{R}^2$? What is the matrix for T using the basis $((1, 1), (1, -1))$? What is the "canonical matrix" for T? By "canonical matrix", I mean the form given in the first problem.

Z3. Verify that $(1, 2, 2)$ is a solution to the equations:

$$\begin{array}{ccccccc} 5x_1 & - & 2x_2 & + & 2x_3 & = & 5 \\ 4x_1 & - & 3x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 5, \end{array}$$

and then find all possible solutions. If there are no more, say so.

Z4. Let $T \in \mathcal{L}(F_3[x])$ have matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}$$

with respect to the basis $(1, x, x^2, x^3)$ for polynomials of degree ≤ 3. Let $v = x^3$, and define a new basis by (v, Dv, D^2v, D^3v), where D is the derivative operator. (You should satisfy yourself that this is indeed a basis.) What is the matrix of T with respect to this new basis?

22. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

    COMMENT: If ST is invertible, all you know is that there is some operator, R, such that $R(ST) = (ST)(R) = I$.

23. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

    Hint: Use the result of the previous problem.

24. Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

    Hint: This problem can be a bit tricky. There are many ways to solve it. The matrix equivalent is that if A is an $n \times n$ matrix then $AB = BA$ for all $n \times n$ matrices B if and only if $A = cI$ for some scalar c, where I is the $n \times n$ identity matrix. Consider the implications of $E_{ij}A = AE_{ij}$ for **all** E_{ij}, where E_{ij} is the elementary $n \times n$ matrix containing 1 in row i and column j and 0 everywhere else. Use the handout (available on the course web site).

25. Prove that if V is finite dimensional with $\dim V > 1$, then the set of non-invertible operators on V is not a subspace of $\mathcal{L}(V)$.

    You might wish to prove this in terms of matrices. No matter how you prove it, all you need is a single counterexample!

26. Not assigned. I believe that I dealt with this sufficiently in class. If you think otherwise, please let me know.