# Problem Set 3

## Chapter 3

Note: Z1, Z2 etc. refer to my own problems, which may be very close to what is in the text.

Z1. Let $V = \mathbb{F}_6[x]$ and let $D$ be the derivative operator. That is, $D \in \mathcal{L}(\mathbb{F}_6[x])$. Compute a basis for $U = \operatorname{null} D^3 \cap \operatorname{range} D^3$. What is the dimension of $U$?

Z2. Make no assumptions about the dimension of $V$. Let $T \in \mathcal{L}(V)$. $n$ is a positive integer.

1. Prove that $\operatorname{null} T^n \subseteq \operatorname{null} T^{n+1}$ and that $\operatorname{range} T^{n+1} \subseteq \operatorname{range} T^n$. In words, if you keep applying the same operator, the null space can only stay the same or become larger, and the range can only stay the same or become smaller.

2. Prove that $T$ is surjective if and only if $T^n$ is surjective and that $T$ is injective if and only if $T^n$ is injective.

Z3. Let $T \in \mathcal{L}(V)$ and suppose that $\operatorname{null} T = \operatorname{range} T$. Prove that $T^2 = 0$.

Z4. Let $V = \mathbb{F}^\infty[x]$. $V$ is, of course, infinite dimensional. Give an example of $T \in \mathcal{L}(V)$ such that $\operatorname{null} T = \operatorname{range} T$, $\dim \operatorname{null} T = \infty$.

Z5. Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\dim(\operatorname{null} T \cap \operatorname{range} T) = 2$ and $\dim(\operatorname{null} T + \operatorname{range} T) = 2$.

Z6. Let $S_B \in \mathcal{L}(\mathbb{F}^\infty)$ be the back shift operator. That is, $S_B(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots)$. Compute the null space of $S_B + 5I$, where $I$ is the identity operator. Is it finite dimensional?

5. Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \dots, v_n)$ is linearly independent in $V$. Prove that $(Tv_1, \dots, Tv_n)$ is linearly independent in $W$.

10. Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

11. Prove that if there exists a linear map on $V$ whose null space and range are both finite dimensional, then $V$ is finite dimensional. You cannot assume

that $V$ is finite dimensional, so you cannot use theorem 3.4. Solving this problem shows that you really understand the proof of theorem 3.4.

14. Solve 14 or 15 (one out of two). This problem shows under what conditions a “left inverse” of a linear map must exist.

15. Solve 15 or 14 (one out of two). This problem shows under what conditions a “right inverse” of a linear map must exist.

16. This problem is harder than most. For this reason, I give a hint. Suppose that $U$ and $V$ are finite-dimensional vector spaces and that $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

Hint: If $T \in \mathcal{L}(U, W)$ and $U_1$ is a subspace of $U$, then $T$ is also a linear map on $U_1$. To be “correct”, we can define $T_1 \in \mathcal{L}(U_1, W)$ by $T_1(u) = T(u)$ for $u \in U_1$. Then $\dim U_1 = \dim \operatorname{null} T_1 + \dim \operatorname{range} T_1$. Apply this idea to solve the problem.