Problem Set 2

Chapter 2

Note: Z1, Z2 etc. refer to my own problems, which may be very close to what is in the text. Please note that despite all the writing, only seven problems have been assigned.

1. Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

You may solve this problem directly, or you may wish to take advantage of something I proved in class. I showed that $(v_1, v_2, \ldots, v_n)$ and $(v'_1, v'_2, \ldots, v'_n)$ have the same span, where $v'_i = v_i$ for $i \neq j$, and $v'_j = v_j + av_k$ for any scalar a and $k \neq j$.

3. Suppose $(v_1, \ldots, v_n)$ is linearly independent in V and $w \in V$. Prove that if $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \text{span}(v_1, \ldots, v_n)$.

Note: You can solve problem 1 with "no sweat". This problem will require some writing.

4. Not assigned. I gave this one to you already and the answer is NO.

5. Not assigned. In class, I showed that the vector space of polynomials is infinite dimensional because if it were spanned by a finite list of polynomials, then x^{m+1} would not be in the span, where m is the maximum degree of the basis polynomials. This means that the vector space of power series is infinite dimensional, because the subspace of polynomials is infinite dimensional.

I did not define $\mathbb{F}^\infty$ in class, but defined power series instead. They are more important.

$\mathbb{F}^\infty := \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{F} \; \forall \, i\}$.

How does the proof that power series are an infinite dimensional vector space apply to $\mathbb{F}^\infty$? We define a subspace of $\mathbb{F}^\infty$ that is analogous to polynomials. Let V be the subspace of $\mathbb{F}^\infty$ such that if $(a_1, a_2, a_3, \ldots) \in V$, then there is some positive integer m such that $a_i = 0 \quad \forall \quad i > m$. V is to $\mathbb{F}^\infty$ as polynomials are to power series. The same kind of proof applies.

You can also use Theorem 2.6 (see the margin comment).

Z8. Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + 5x_2 = 0 \quad \text{and} \quad 2x_3 - 6x_4 = 0\}.$$

Find a basis of U.

Z9a. Prove or disprove: If $0 \leq k < m$, where $k, m \in \mathbb{Z}$, there exists a basis, $(p_0(x), p_1(x), \ldots, p_m(x))$ of $\mathbb{F}[x]$ such that none of the basis polynomials has degree k. If true, write down a basis with this property.

Z9b. Prove or disprove: If $0 \leq m$, there exists a basis $(p_0(x), p_1(x), \ldots, p_m(x))$ of $\mathbb{F}[x]$ such that all basis polynomials have the same degree. If true, write down a basis with this property.

10. Not assigned. This was covered in class.

13. Suppose U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $U \cap W = \{0\}$.

14. Suppose that U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

16. Not assigned. Covered in class.

17. Not assigned. Covered in class.