Problem Set 1

Chapter 1

Note: Z1, Z2 etc. refer to my own problems, which may sometimes be similar to what is in the text.

Z1. $V = \mathbb{R}[x]$ (polynomials with real coefficients). True or False. In each case, assume that the zero polynomial is included in the set. Give a counterexample when your answer is False. Please do not give the zero polynomial as a counterexample.

- Are polynomials of degree 20 a subspace of V?
- Are polynomials of degree ≤ 20 a subspace of V?
- Are polynomials of degree ≥ 20 a subspace of V?
- Are polynomials of odd degree a subspace of V?
- Are polynomials where $a_i = 0$ if i is not a multiple of 7 or 13 a subspace of V?
- Are polynomials where $a_i = 0$ if i is prime number a subspace of V?

Z2. Let $V = \mathbb{R}[x]$. Give an example of two non-zero subspaces of V, U_1 and U_2, where neither is a subspace of the other and $U = U_1 + U_2$ is not a direct sum. (Hint: Using proposition 1.9 might make the problem easier.)

6. Give an example of a nonempty subset, U, of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

7. Give an example of a nonempty, subset U, of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

8. Prove that the intersection of any collection of subspaces of V is a subspace of V. To be specific, let $\{U_\alpha\}_{\alpha\in\Gamma}$ be a collection of subspaces of V, Γ is an arbitrary index set.

   For this course, it is not so critical to use this kind of generality. It would be sufficient to use, for example, U_i, for $i = 1, 2, 3 \ldots$. In fact, the proof is the same as long as there are at least two subspaces.

10. Suppose that U is a subspace of V. What is $U + U$? What is aU, where a is a non-zero scalar? (Note: $aU = \{u \in U \ : \ u = au' \quad \text{for some} \quad u' \in U\}$.)

13. Prove or give a counterexample: If U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$. Do not give a trivial example where one or more of the subspaces is $\{0\}$.

15. Prove or give a counterexample: If U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.