

Elementary Linear Maps

Let $T \in \mathcal{L}(V, W)$, where $\dim V = n$ and $\dim W = m$. Furthermore, let (v_1, v_2, \dots, v_n) and (w_1, w_2, \dots, w_m) be bases for V and W , respectively. Let $\mathcal{M}(T) = A$ with respect to the two bases. The scalars are \mathbb{F} .

For any i and j , ($1 \leq i \leq m$ and $1 \leq j \leq n$), let $\Delta_{i,j}$ be the linear map defined by

$$\begin{aligned}\Delta_{i,j}v_k &= 0 & k \neq j \\ \Delta_{i,j}v_j &= w_i & (k = j).\end{aligned}$$

$\dim \text{null } \Delta_{i,j} = n - 1$, and $\text{null } \Delta_{i,j}$ is spanned by all of the v_k vectors **except** for v_j , which is mapped to w_i and generates $\text{range } \Delta_{i,j}$.

Let $E_{i,j} = \mathcal{M}(\Delta_{i,j})$. Then row i and column j of $E_{i,j}$ contains 1. All other entries are 0.

The vector space of linear maps from V to W is isomorphic to the vector space of $m \times n$ matrices with entries in \mathbb{F} . In fact, the isomorphism can be defined by mapping $\Delta_{i,j}$ to $E_{i,j}$. These $m \times n$ linear operators (or matrices) are linearly independent and span $\{T : V \rightarrow W \mid T \in \mathcal{L}(V, W)\}$ (or $\text{Mat}(m, n, \mathbb{F})$).

Elementary linear operators

Now let $T \in \mathcal{L}(V)$, and let (v_1, v_2, \dots, v_n) be a basis for V . Define A , $\Delta_{i,j}$ and $E_{i,j}$ as above.

$\Delta_{i,j}T$ *versus* $T\Delta_{i,j}$

What are $\Delta_{i,j}T$ and $T\Delta_{i,j}$, and how do they compare?

$$\begin{aligned}\Delta_{i,j}Tv_k &= \sum_{l=1}^n a_{l,k} \Delta_{i,j}v_l \\ &= a_{j,k}v_i\end{aligned}$$

On the other hand,

$$\begin{aligned}
 T\Delta_{i,j}v_k &= T0 \\
 &= 0 && \text{if } k \neq j, \text{ and} \\
 T\Delta_{i,j}v_j &= Tv_i \\
 &= \sum_{l=1}^n a_{l,i}v_l \quad (k = j).
 \end{aligned}$$

E_{i,j}A versus AE_{i,j}

What is the matrix equivalent? First note that

$$E_{i,j} = \begin{matrix} & & & 1 & 2 & \dots & j & \dots & n \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{matrix} & \left(\begin{matrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{matrix} \right). \end{matrix}$$

Translating $\Delta_{i,j}T$ into matrix form yields

$$E_{i,j}A = \begin{matrix} & & & 1 & 2 & \dots & j & \dots & n \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{matrix} & \left(\begin{matrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j,1} & a_{j,2} & \dots & a_{j,j} & \dots & a_{j,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{matrix} \right), \end{matrix}$$

and translating $T\Delta_{i,j}$ into matrix form yields

$$AE_{i,j} = \begin{matrix} & & & 1 & 2 & \dots & j & \dots & n \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{matrix} & \left(\begin{matrix} 0 & 0 & \dots & a_{1,i} & \dots & 0 \\ 0 & 0 & \dots & a_{2,i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{i,i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,i} & \dots & 0 \end{matrix} \right). \end{matrix}$$

In words, multiplying A on the left with $E_{i,j}$ creates a matrix whose i^{th} row is the j^{th} row of A . All other rows contain only 0s. Multiplying A on the right with $E_{i,j}$ creates a matrix whose j^{th} column is the i^{th} column of A . All other columns contain only 0.