Change of basis example

. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be defined by its matrix (standard basis),

$$\mathcal{M}(T) = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

What is $\mathcal{M}(T)$ in terms of the basis, $((1,1,1),(1,1,0),(1,0,0)$?

Solution: . Using the matrix, note that $T(x,y,z) = (2x - z, x + y + z, x - y)$. The "direct" solution is to compute,

- $T(1,1,1) = (1,3,0) = 3(1,1,0) - 2(1,0,0)$,
- $T(1,1,0) = (2,2,0) = 2(1,1,0)$ and
- $T(1,0,0) = (2,1,1) = (1,1,1) + (1,0,0)$.

Expressing $T(1,1,1)$, $T(1,1,0)$ and $T(1,0,0)$ as linear combinations of $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$ is done by inspection. From the above,

$$\mathcal{M}(T, ((1,1,1),(1,1,0),(1,0,0)) = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

What is the formal approach?
Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

the matrix of T with respect to the standard basis of $\mathbb{R}^3$, (e_1, e_2, e_3). Define $(v_1, v_2, v_3) = ((1,1,1),(1,1,0),(1,0,0))$. Let

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then B is a "change of basis matrix". More precisely, B is the matrix of the identity operator with respect to (v_1, v_2, v_3) and (e_1, e_2, e_3). This matrix contains information to go from the (v_1, v_2, v_3) basis to the standard basis. Column 1 contains v_1 expressed as $1e_1 + 1e_2 + 1e_3$, and so on. To convert the other way, we need to invert B. This can be done, for example, by a succession of elementary row transformations performed simultaneously on B and I, until B is converted to I. These elementary transformations are the equivalent of mutplying on the left by $I + rE_{i,j}$, which adds r times the j^{th} row to the i^{th} row. These elementary transformations are clearly invertible. One can also "swap rows".

The process is:

Begin:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtract row 3 from rows 1 and 2.

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtract row 2 from row 1.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Swap rows 1 and 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Thus,

$$B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

The matrix for T with respect to the new basis is $B^{-1}AB$.

$$
\begin{aligned}
B^{-1}AB &= B^{-1}\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

This is the formal way to derive a matrix with respect to a new basis from a given matrix for an initial basis.

Much of the course is devoted to choosing a basis so that $B^{-1}AB$ has a simple form, such as “upper triangular” or “diagonal”.